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H-SPACES OF CATEGORY ≤ 2†

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§0. INTRODUCTION

IN [1] ADAMS completed the answer to the question of which spheres admit multiplications (continuous with identity). In [2] he noted that by generalizing spheres slightly, one gets new H-spaces with interesting properties. These new H-spaces, like the spheres, are suspensions. The purpose of this paper is to classify all suspensions (in fact, all spaces of category ≤ 2) which admit multiplications. The second part of the paper is devoted to the question of which suspensions have the homotopy type of a loop space (such as the three-dimensional sphere).

Note. All spaces are assumed to have the homotopy type of connected CW complexes.

§1

Following [2], we let $Y(G, n)$ denote the Moore space of type (G, n), i.e. a simply-connected space with trivial homology except for $H_n = G$. Also in [2] is the proof of the following lemma (as part of the theorem on page 54).

LEMMA 1. *If $n \geq 3$ and G is a subgroup of the group Q of rational numbers such that $2G = G$, then $Y(G, n)$ is an H-space. If $n = 3$ or 7, the result holds even if $2G \neq G$.*

Using this lemma, together with the fact that for n odd, $Y(Q, n)$ is the same as the Eilenberg–MacLane space $K(Q, n)$ (also noted in [2]), we can prove the following lemma (in which $Z(p) = Q/Q_p$, $Q_p = \{r/s \in Q : s \text{ is prime to } p\}$).

LEMMA 2. *If $n \geq 2$ is even and p is odd, then $Y(Z(p), n)$ is an H-space. If p is even, the result holds for $n = 2$ or 6.*

Proof. Realize the inclusion $Q_p \subset Q$ by a map $Y(Q_p, n+1) \to Y(Q, n+1)$. Turn this into a fibration. From the spectral sequence it follows that the fiber is $Y(Z(p), n)$. In fact, it is clear that the fiber is correct at least up to dimension n. Since $H_i(Y(Q, n+1); Z(p)) = 0$ for all $i > 0$, no other homology arises. Now it follows from the commutative

† The first part of this paper is essentially contained in a thesis completed in 1964 at Harvard under the direction of R. Bott.

diagram

$$\begin{array}{ccc} Y(Z(p),n)\vee Y(Z(p),n) & \longrightarrow & Y(Z(p),n) \\ \downarrow & & \downarrow \\ Y(Z(p),n)\times Y(Z(p),n) \to Y(Q_p,n+1)\times Y(Q_p,n+1) & \to & Y(Q_p,n+1) \end{array}$$

that the obstructions to restricting a multiplication on $Y(Q_p, n+1)$ (which exists by Lemma 1) to $Y(Z(p), n)$ lie in relative cohomology groups of the pair on the left with coefficients in the relative homotopy of the pair on the right. But this relative homotopy is the homotopy of $Y(Q, n+1) = K(Q, n+1)$, by the fibration discussed above. By the Künneth formula and universal coefficient theorem, these cohomology groups vanish, yielding the existence of the desired multiplication.

Now suppose X is a space of category ≤ 2 with a multiplication. By the restriction on cat X, the cohomology over any ring has no non-trivial products. On the other hand, the existence of a multiplication puts a Hopf algebra structure on the cohomology over any field. These two facts imply, by [7], that for any field F, there is at most one integer $i > 0$ with $H^i(X, F) \neq 0$. Also, if there is such an i, then $H^i(X; F) = F$ and if char $F \neq 2$ then i is odd. If char $F = 2$, then [1] implies $i = 1$, 3, or 7.

THEOREM 1. *An H-space of category ≤ 2 is either simply connected or homotopy equivalent to the circle S^1.*

Proof. Let X be such a space and assume $\pi_1(X) \neq 0$. Since X is an H-space $\pi_1(X)$ is abelian, but by [4], cat $X \leq 2$ implies $\pi_1(X)$ is free. The only group satisfying these three conditions is Z. It follows then that $H_1(X) = Z$ and $H^1(X; F) = F$ for any field F. By the preceding discussion $H^i(X; F) = 0$ for any $i > 1$ and any field F. The universal coefficient theorem now implies that X has the homology of S^1. Since S^1 is a $K(Z, 1)$, there is a map $X \to S^1$ inducing isomorphisms on π_1 and H_*. This map turns out to be the required homotopy equivalence, as can be seen by turning it into fibration and considering the fiber W. What we need is to show that all homotopy groups of W vanish. Since $\pi_1(W) = 0$ it will suffice to show that W has trivial homology, and this in turn will follow from the spectral sequence if we can show that $\pi_1(S^1)$ acts trivially on $H_*(W)$. But if l is a loop representing the generator of $\pi_1(S^1)$ there is a loop $\bar{l}$ in X covering it and (assuming the base-point of X is its multiplicative identity) $(w, t) \to w \cdot \bar{l}(t)$ is a homotopy of W into X which covers l and starts at the inclusion $W \subset X$. Since it also ends at the inclusion, it follows that l acts trivially.

THEOREM 2. *Suppose given*

(1) *A subgroup $G \subset Q$ of the rational numbers (G may be O),*

(2) *An odd integer $n \geq 3$, with n either* 3 *or* 7 *if* $2G \neq G$,

(3) *A set P of primes such that $pG = G$ for each $p \in P$,*

(4) *For each $p \in P$ an even integer $n_p \geq 2$, with $n_2 = 2$ or 6 if $2 \in P$.*

Then $Y(G, n) \vee V_{p \in P}\, Y(Z(p), n_p)$ is an H-space (as well, of course, as a suspension). These spaces are, up to homotopy, the only simply-connected H-spaces of category ≤ 2.

Proof. Since each of the spaces in the indicated one-point union is an H-space, the first statement follows from the remark that the union may be replaced by the corresponding (weak) product without affecting homotopy type, since the tensor and tor terms of the Künneth formula all vanish in this case.

Conversely, if X is a simply-connected H-space of category ≤ 2, the discussion of the cohomology of X at the beginning of this section shows that X has the correct cohomology over any field to be of the indicated type. In fact, by the universal coefficient theorem, it must have the right (integral) homology. To show that it is actually of the desired homotopy type, we must realize the various factors of $H_*(X)$ by maps from the corresponding Moore spaces into X. If the homology is all torsion (excluding H_0, of course), the possibility of finding these maps follows from Serre's generalized Hurewicz theorem [8]. If the torsion-free part is non-trivial, then the homology of X consists of a subgroup G of Q in some odd dimension n and various $Z(p)$'s scattered among the even dimensions. Let $f\colon X \to K(Q, n)$ be a map which on H_n is the inclusion $G \to Q$. Turning it into a fibration and considering the spectral sequence, we see that the fiber W has the same homology as X except that H_n has disappeared and Q/G has been added to H_{n-1}. Since Q/G is a direct sum of $Z(p)$'s for primes not already in the torsion of $H_*(X)$, it follows that W is the sort of space for which we have already realized the homology by maps of Moore spaces. Since all the torsion in $H_*(X)$ comes from $H_*(W)$, it follows that the torsion can be realized. All that remains is G in dimension n. The problem is to lift through f the map $Y(G, n) \to Y(Q, n) = K(Q, n)$ corresponding to the inclusion $G \to Q$. The obstructions to constructing such a lift lie in $H^i(Y(G, n); \pi_{i-1}(W))$ for $i = n, n + 1$.

To see that these obstructions vanish we must consider the homotopy of $Y(Z(p), j)$ for even j. Since the fiber of an essential map $S^{j+1} \to K(Q, j + 1)$ is the one-point union (or equivalently, the weak product) of the spaces $Y(Z(p), j)$, p varying over all primes, it follows that $\pi_{j+1}(Y(Z(p), j)) = 0$, while $\pi_{j+k}(Y(Z(p, j))$ is the p-primary part of $\pi_{j+k}(S^{j+1})$ for $k > 1$. The kernel of the Hurewicz map $\pi_{n-1}(W) \to H_{n-1}(W)$ is therefore a finite group whose order r satisfies $rG = G$. Thus, a map $G \to \pi_{n-1}(W)$ vanishes if its composition with the Hurewicz map vanishes. In particular, the restriction to G of the homotopy boundary operator ∂ vanishes since its composition with the Hurewicz map is the homology transgression, which certainly vanishes on G. But $\partial | G \in \mathrm{Hom}(G, \pi_{n-1}(W)) = H^n(Y(G, n); \pi_{n-1}(W))$ is the first obstruction. Next, since $\pi_n(Y(Q/G, n - 1)) = 0$, $\pi_n(W)$ is a finite group of order s with $sG = G$. Consequently, $H^{n+1}(Y(G, n); \pi_n(W)) = \mathrm{Ext}(G, \pi_n(W))$ vanishes.

§2

Following the terminology of Theorem 2, let $X = Y(G, n) \vee V_{p \in P}\, Y(Z(p), n_p)$ be an H-space of category ≤ 2. If X has the homotopy type of a loop space ΩB, then for any $p \in P$, $H^*(B; Z_p)$ is a polynomial ring on one generator in dimension $n_p + 2$, while for any prime p with $pG \neq G$, $H^*(B; Z_p)$ is a polynomial ring on one generator in dimension $n + 1$. By [6], it follows that for every odd prime $p \in P$, $n_p + 2$ divides $2(p - 1)$ and for any odd prime p with $pG \neq G$, $n + 1$ divides $2(p - 1)$. For $p = 2$, Theorem 2 of [9] implies that $n_2 = 2$ if $2 \in P$ while $n = 3$ if $2G \neq G$. These necessary conditions are also sufficient for the existence of B. The first step toward proving this will be the case $X = Y(Z(p), n)$.

The endomorphism ring of $Z(p)$ is isomorphic to the ring of p-adic integers, in which there is a primitive $p-1$ root ξ of the identity. Then for any divisor k of $p-1$, say $kl = p-1$, $\eta = \xi^l$ is an automorphism of $Z(p)$ of order k. This defines an action of Z_k on $Z(p)$ and allows us to form the semi-direct product $S = S_{p,k} = Z(p) \underset{\eta}{\times} Z_k$, which contains $Z(p)$ as a normal subgroup and Z_k as a subgroup whose generator, acting by conjugation on $Z(p)$, acts like η. Theorem 10.1 of [3], and the fact that homology of groups commute with direct limits, implies that the homology of $K(S, 1)$ is all torsion (except of course H_0) and that its p-primary component consists of one copy of $Z(p)$ in each dimension of the form $2ik-1$ ($i = 1, 2, \ldots$), while its cohomology over Z_p is a polynomial ring on one generator in dimension $2k$. If we could form a simply connected space B and a map $S \to B$ mapping the p-primary part of $H_*(S)$ isomorphically onto $H_*(B)$, it would follow that ΩB is of type $Y(Z(p), 2k-2)$, completing the first step.

LEMMA 3. *There exists a simply connected space B and a map $K(S, 1) \to B$ mapping the p-primary part of $H_*(K(S, 1))$ isomorphically onto $H_*(B)$.*

Proof. We construct a sequence of approximations to B. Let $K = K(S, 1)$, $H_n = H_n(K)$, $H_n^{(p)} = p$-primary part of H_n. Let B_1 = point, $K \to B_1$ the unique map. Assume we have already constructed $K \to B_j$ such that for $n \leq j$, $H_n(B_j) = H_n^{(p)}$ and $H_n(K) \to H_n(B_j)$ is the canonical projection and such that $H_{j+1}^{(p)}$ maps onto $H_{j+1}(B_j)$. Construct $K \to B_{j+1}$ as follows: Turn $K \to B_j$ into a fibration with fiber F_j. $H_n(F_j)$ has trivial p-component for $n \leq j$. Let $J = J_{j+1} = H_{j+1}^{(p)}(F_j)$. The projection $H_{j+1}(F_j) \to J$ represents an element $\sigma \in H^{j+1}(F_j; J) = \mathrm{Hom}(H_{j+1}(F_j), J)$. Let $\tau \in H^{j+2}(B_j; J)$ be the transgression of σ. Then τ defines a fibration $K(J, j+1) \to B_{j+1} \to B_j$. $K \to B_j$ lifts through B_{j+1} since τ pulls back to 0 in $H^{j+2}(K; J)$. We can assume that this map $K \to B_{j+1}$ (which may not yet be the map we want) is a fiber map over B_j. The induced map $F_j \to K(J, j+1)$ represents an element $\bar{\sigma} \in H^{j+1}(F_j; J)$. Although $\bar{\sigma}$ may differ from σ, its transgression must be τ (since τ is the characteristic class of the fibration $K(K, j+1) \to B_{j+1} \to B_j$). Thus $\sigma - \bar{\sigma}$ has transgression 0, and so extends over K, defining a map $K \to K(J, j+1)$. Combining this map with the original map $K \to B_{j+1}$ via the operation $K(J, j+1) \times B_{j+1} \to B_{j+1}$ gives us a map $K \to B_{j+1}$ which (by comparison of spectral sequences) has the desired properties. By construction we have an inverse system $B_1 \leftarrow B_2 \leftarrow \ldots$ together with a compatible sequence of maps $K \to B_j$. Since $B_{j+1} \to B_j$ is a homotopy equivalence up to dimension j, we can successively modify the B_j's so that $B_{j+1} \to B_j$ actually maps the j-skeleton of B_{j+1} homeomorphically onto that of B_j. Forming the obvious limit complex, we get the required $X \to B$.

COROLLARY. *If k divides $p-1$ ($k > 1$), then $Y(Z(p), 2k-2)$ has the homotopy type of a loop space.*

Proof. By spectral sequence argument, ΩB (B as in Lemma 3) is of type $Y(Z(p), 2k-2)$.

By the argument at the beginning of this section, these are the only values of n for which $Y(Z(p), n)$ can be a loop space, provided p is an odd prime. For $p = 2$, one has the following

Remark. $Y(Z(2), 2)$ has the homotopy type of a loop space.

Proof. Take an essential map from infinite quaternionic projective space to $K(Q, 4)$. Use the method of the proof of Lemma 3 to split off the 2-primary part of the fiber of this map. This gives a space whose loop space is of type $Y(Z(2), 2)$.

The next step is to treat the spaces $Y(G, n)$ for $G \subset Q$.

LEMMA 4. *If $G \subset Q$ then $Y(G, n-1)$ has the homotopy type of a loop space if either $n = 4$ or if n is an even divisor of $2(p-1)$ for any prime p with $pG \neq G$.*

Proof. Let P be the set of primes with $pG \neq G$, and let $Z(P)$ be the direct sum $\sum_{p \in P} Z(p)$. Then $Y(Z(P), n-1)$ has the homotopy type of a loop space. Indeed, if $Y(Z(p), n-2) = \Omega(B^{(p)})$, then the weak direct product $\prod_{p \in P}^{w} B^{(p)}$ has as loop space $\prod_{p \in P}^{w} Y(Z(p), n-2)$, which is homotopy equivalent to $\bigvee_{p \in P} Y(Z(p), n-2) = Y(Z(P), n-2)$. Since the homology of B consists of copies of $Z(P)$ in dimension $ni-1$, we can form B by starting with a point and successively attaching cones over $Y(Z(P), ni-2)$. The proof will be accomplished by showing that these can be replaced by cones over $Y(G^{\otimes i}, ni-1)$ ($G^{\otimes i} = i$-th tensor power). Note first that since $Q/G \cong Z(P)$, the fiber of any essential map $Y(G, n) \to Y(Q, n) = K(Q, n)$ is $Y(Z(P), n-1)$. We thus get a map $Y(Z(P), n-1) \to Y(G, n)$ which induces isomorphisms on mod p cohomology for all primes p. Let $B_j \subset B$ be the nj-th homology section (i.e. B_j is formed, as B, by attaching cones over $Y(Z(P), ni-2)$, but the process stops at $i = j$). Then $B_1 = Y(Z(P), n-1)$. Let $C_1 = Y(G, n)$, $f_1 : B_1 \to C_1$ the map constructed above. Suppose, inductively, that we have constructed a space C_{j-1} by attaching cones over $Y(G^{\otimes i}, ni-1)$ and a map $f_{j-1} : B_{j-1} \to C_{j-1}$ inducing isomorphisms on mod p cohomology for all p. Let us now compute some homotopy groups of C_{j-1}. We do this by comparing with $K(G, n)$. Using spectral sequence computations of the homology and the mod p cohomology rings of $K(G, n)$ (starting with the fact that $K(G, 1)$ has no homology in dimensions above one), one finds that a map $C_{j-1} \to K(G, n)$ which induces the identity on $H_n = G$ must induce isomorphisms modulo the class $\mathscr{C}$ of finite P-groups on homology up to dimension $nj-1$. Since $H_{nj}(K(G, n))$ is isomorphic modulo $\mathscr{C}$ to $G^{\otimes j}$, so is $\pi_{nj-1}(C_{j-1})$. Since any subgroup of $G^{\otimes j}$ with finite index is isomorphic to $G^{\otimes j}$ ($G^{\otimes j}$ being torsion-free of rank one), $\pi_{nj-1}(C_{j-1}) \cong G^{\otimes j} \oplus F$, where F is a finite P-group. Also $\pi_{nj-2}(C_{j-1})$ is a finite P-group. By the universal coefficient theorem for homotopy [5], $[Y(Z(P), nj-2), C_{j-1}] \cong \mathrm{Ext}(Z(P), G^{\otimes j}) \oplus \mathrm{Ext}(Z(P), F) \cong \mathrm{Ext}(Z(P), G^{\otimes j}) \oplus F$, $[Y(Q, nj-2), C_{j-1}] \cong \mathrm{Ext}(Q, G^{\otimes j})$. Now, let $\zeta \in H_n(C_{j-1}; G)$. be the element corresponding to the identity in $\mathrm{Hom}(G, G) \cong \mathrm{Hom}(H_n(C_{j-1}), G) \cong H^n(C_{j-1}; G)$. For any $g \in [Y(Z(P), nj-2), C_{j-1}]$ we get a functional j-th power map $H^n(C_{j-1}; G) \to H^{nj-1}(Z(P); G^{\otimes j})$. Evaluation on ζ gives a homomorphism $[Y(Z(P), nj-2), C_{j-1}] \to H^{nj-1}(Z(P), G^{\otimes j})$. By the assumptions on the cohomology ring of B, the image under this homomorphism of $f_{j-1} \circ \alpha$ ($\alpha : Y(Z(P), nj-2) \to B_{j-1}$ the attaching map for B_j) is not divisible by any $p \in P$. Since $H^{nj-1}(Z(P), G^{\otimes j})$ is torsion-free (it is the product of the p-adic integers, p varying through P), the finite summand of $[Y(Z(P), nj-2), C_{j-1}]$ is annihilated by the above homomorphism, and so the component of $f_{j-1} \circ \alpha$ in $\mathrm{Ext}(Z(P), G^{\otimes j})$ is not divisible by any $p \in P$. This is equivalent to saying that in the associated extension $0 \to G^{\otimes j} \to E \to Z(P) \to 0$, $E \cong Q$. If this sequence is used to define a cofibration $Y(Q, nj-2) \to Y(Z(P), nj-2) \to Y(G^{\otimes j}, nj-1)$, then $f_{j-1} \circ \alpha$ pulls back to zero on $Y(Q, nj-2)$ and therefore extends over $Y(G^{\otimes j}, nj-1)$. We get a commutative diagram

$$\begin{array}{ccc} Y(Z(P), nj-2) & \rightarrow & Y(G^{\otimes j}, nj-1) \\ \downarrow & & \downarrow \\ B_{j-1} & \rightarrow & C_{j-1} \end{array}$$

Taking mapping cones of the vertical arrows gives $f_j : B_j \rightarrow C_j$, allowing us to continue the induction. Letting C be the union of the C_j's (with the weak topology), we get ΩC of type $Y(G, n-1)$.

THEOREM 3. *A space $Y(G, n) \vee V_{p \in P} Y(Z(p), n_p)$ has the homotopy type of a loop space if and only if the following three conditions are satisfied:*

(1) *n is odd and $n+1$ divides $2(p-1)$ for each p with $pG \neq G$ or else $n = 3$,*
(2) *n_p is even and $n_p + 2$ divides $2(p-1)$ for each $p \in P$ or else $n_p = 2$,*
(3) *$pG = G$ for each $p \in P$.*

Furthermore, these are the only simply-connected loop spaces of category ≤ 2.

Proof. All that requires comment is that the one-point union may be replaced by the weak cartesian product without changing homotopy type.

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